

COMPUTING DEGREE AND CLASS DEGREE

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ABSTRACT. Let π be a factor code from a one dimensional shift of finite type X onto an irreducible sofic shift Y . If π is finite-to-one then the number of preimages of a typical point in Y is an invariant called the degree of π . In this paper we present an algorithm to compute this invariant. The generalized notion of the degree when π is not limited to finite-to-one factor codes, is called the class degree of π . The class degree of a code is defined to be the number of transition classes over a typical point of Y and is invariant under topological conjugacy. We show that the class degree is computable.

1. INTRODUCTION

One source of inspiration in symbolic dynamics comes from storage systems and transmission in computer science. For example sofic shifts are analogous to regular languages in automata theory, so a sofic shift and its cover are natural models for information storage and transmission. As a result, starting with a presentation of a dynamical system, there are known algorithms constructed to compute some kind of object from such presentation. Given a sofic shift, Coven and Paul constructed a finite procedure to obtain a finite-to-one sofic cover [5]. There is an algorithm to determine whether two graphs present the same sofic shift [8]. Kim and Roush showed that the shift equivalence of sofic systems is decidable [7].

In this work, starting from a sofic shift and its finite-to-one cover, we present an algorithm to compute the number of preimages of a typical point of the sofic. Moreover, we show that in the case of having an infinite-to-one cover, an analogous object can be computed in finitely many steps.

Given a factor code π from a one-dimensional shift of finite type X to a sofic shift Y , when π is finite-to-one there is a quantity assigned to π called the *degree* of π . The degree of a finite-to-one code is defined to be

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the minimal number of π -preimages of the points in Y . One can show that the number of preimages of every transitive point in Y is exactly the degree of π . The degree of a finite-to-one code is widely-studied and known to be invariant under recoding [8]. In the first section of this paper we present an algorithm to compute this invariant.

When $\pi : X \rightarrow Y$ is not limited to be finite-to-one an analogous of the degree, called the *class degree*, is defined to be the minimal number of transition classes (always finite) over the points in Y . The definition of a transition class is motivated by communicating classes in Markov chains. Roughly speaking, two preimages x and \bar{x} of a point y in Y lie in the same equivalence class, *transition class*, if one can find a preimage z of y which is equal to x up to an arbitrarily large given positive coordinate and right asymptotic to \bar{x} and vice versa. When π is finite-to-one then the degree and the class degree of π match. One can also show that the class degree is invariant under topological conjugacy and the number of transition classes over any transitive point of Y is exactly the class degree of π . One of the main applications of the class degree is bounding the number of measures of relative maximal entropy [1]. Such measures have applications in information theory, computing Hausdorff dimensions and functions of Markov chains [2, 3, 4, 6, 9]. In the second section of this paper we show that the class degree is computable.

2. BACKGROUND

Throughout this paper, X is a one-dimensional shift of finite type (SFT) with the shift transformation T . The alphabet of X is denoted by $\mathcal{A}(X)$ and the σ -algebra on X generated by cylinder sets is denoted by \mathcal{B}_X . A triple (X, Y, π) is called a **factor triple** when $\pi : X \rightarrow Y$ is a continuous shift-commuting map (factor code) from an SFT X onto a subshift Y (sofic shift Y). When π is a one-block factor code induced by a symbol-to-symbol map $\pi_b : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)$ we naturally extend π_b to blocks in \mathcal{B}_X (b stands for block). When π is a finite-to-one factor code there is a uniform upper bound on the number of pre-images of points in Y . The minimal number of pre-images of points in Y is called the **degree** of the code and is denoted by d_π .

Definition 2.1. We say two factor triples (X, Y, π) and $(\tilde{X}, \tilde{Y}, \tilde{\pi})$ are **conjugate** if X is conjugate to \tilde{X} under a conjugacy ϕ , Y is conjugate to \tilde{Y} under a conjugacy ψ , and $\tilde{\pi} \circ \phi = \psi \circ \pi$.

Theorem 2.2. [8] Let (X, Y, π) be a factor triple. There is a factor triple $(\tilde{X}, \tilde{Y}, \tilde{\pi})$ conjugate to (X, Y, π) such that \tilde{X} is one-step and $\tilde{\pi}$ is one-block.

Theorem 2.3. [8] *Given two conjugate factor triples (X, Y, π) and $(\tilde{X}, \tilde{Y}, \tilde{\pi})$, we have $d_\pi = d_{\tilde{\pi}}$.*

Theorem 2.4. [8] *Let π be a finite-to-one factor code from an SFT X onto an irreducible sofic shift Y . Then every transitive point of Y has exactly d_π preimages.*

Given a one-block factor code π , above every Y -block W there is a set of X -blocks U which are sent to W by π_b ; i.e., $\pi_b(U) = W$. Given $0 \leq i < |W|$, define

$$\pi_b^{-1}(W)_i = \{a \in \mathcal{A}(X) : \exists W' \text{ with } \pi_b(W') = W, W'_i = a\}$$

and

$$d_\pi^* = \min\{|\pi_b^{-1}(W)_i| : W \in \mathcal{L}(Y), 0 \leq i < |W|\}.$$

Theorem 2.5. [8] *Let π be a finite-to-one one-block factor code from an SFT X onto an irreducible sofic shift Y . Then $d_\pi^* = d_\pi$.*

Given a one-block factor code $\pi : X \rightarrow Y$, a **magic block** is a block W such that $d(W, i) = d_\pi^*$ for some $0 \leq i < |W|$. Such an index i is called a **magic coordinate** of W . A factor code π has a **magic symbol** if there is a magic block of π of length 1.

The class degree defined below is a quantity analogous to the degree which is defined in the general case when π is not only limited to be finite-to-one.

Definition 2.6. *Suppose (X, Y, π) is a factor triple and $x, x' \in X$. There is a **transition** from x to x' denoted by $x \rightarrow x'$ if for each integer n , there is a point v in X so that*

- (1) $\pi(v) = \pi(x) = \pi(x')$, and
- (2) $v_{-\infty}^n = x_{-\infty}^n$, $v_i^\infty = x_i'^\infty$ for some $i \geq n$.

*Write $x \sim x'$, and say x and x' are in the same (equivalence) **transition class** if $x \rightarrow x'$ and $x' \rightarrow x$. The minimal number of transition classes over points of Y is called the **class degree** of π and denoted by c_π .*

Theorem 2.7. [1] *Given two conjugate factor triples (X, Y, π) and $(\tilde{X}, \tilde{Y}, \tilde{\pi})$, we have $c_\pi = c_{\tilde{\pi}}$.*

Theorem 2.8. [1] *Let π be a one-block factor code from a one-step SFT X to a sofic shift Y . The number of transition classes over a right transitive point of Y is exactly the class degree.*

Theorem 2.10, in below, provides a finitary characterization of the class degree.

Definition 2.9. Let $\pi : X \rightarrow Y$ be a one-block factor code from a one-step SFT X to a sofic shift Y and let $W = W_0 \dots W_p$ be a block of Y . Let $0 < n < p$ and let M be a subset of $\pi_b^{-1}(W)_n$. We say $U \in \pi_b^{-1}(W)$ is **routable** through $a \in M$ at time n if there is a block $U' \in \pi_b^{-1}(W)$ with $U'_0 = U_0$, $U'_n = a$, and $U'_p = U_p$. A triple (W, n, M) is called a **transition block** of π if every block in $\pi_b^{-1}(W)$ is routable through a symbol of M at time n . The cardinality of the set M is called the **depth** of the transition block (W, n, M) .

Let

$$c_\pi^* = \min\{|M| : (W, n, M) \text{ is a transition block of } \pi\}.$$

A **minimal transition block** of π is a transition block of depth c_π^* .

Theorem 2.10. [1] Let π be a one-block factor code from a one-step SFT X to a sofic shift Y . Then $c_\pi^* = c_\pi$.

The following theorem shows the relation between the degree and the class degree of a finite-to-one factor code.

Theorem 2.11. [1] Let $\pi : X \rightarrow Y$ be a finite-to-one factor code from a SFT X to an irreducible sofic shift Y . Then $c_\pi = d_\pi$.

3. DEGREE ALGORITHM

In this section we present an algorithm to compute the degree of a finite-to-one factor code. By Theorems 2.2 and 2.3, without loss of generality, we may assume π is a one-block factor code defined on a one-step SFT.

Let X be a one-step SFT with alphabet $\mathcal{A}(X) = \{a_1, \dots, a_i\}$ and adjacency matrix I . Let $\pi : X \rightarrow Y$ be a finite-to-one one-block factor code from X to a sofic shift Y with the alphabet $\mathcal{A}(Y) = \{b_1, \dots, b_j\}$. Make two graphs G and G' as follows.

Let G and G' both have the same vertex set

$$\mathcal{V} = \bigcup_{b \in \mathcal{A}(Y)} \mathcal{P}\{\pi_b^{-1}(b)\} = \{A_1, \dots, A_m\}$$

where $\mathcal{P}\{\pi_b^{-1}(b)\}$ stands for the power set of $\{\pi_b^{-1}(b)\}$. For what we need later, divide the vertex set \mathcal{V} into two parts \mathcal{U} and $\mathcal{V} - \mathcal{U}$ where $\mathcal{U} = \{\pi_b^{-1}(b) : b \in \mathcal{A}(Y)\}$. Form the adjacency matrix M of G and the adjacency matrix M' of G' as follows. Let $A, A' \in \mathcal{V}$, then $A \subseteq \pi_b^{-1}(b)$ and $A' \subseteq \pi_{b'}^{-1}(b')$ for some $b, b' \in \mathcal{A}(Y)$. Say $M_{AA'} = 1$ if the following conditions hold.

- (1) bb' is a block of Y ,

- (2) A' is exactly the set of all symbols a' in $\pi_b^{-1}(b')$ such that aa' is a block of X for some a in A .

Otherwise $M_{AA'} = 0$. Say $M'_{AA'} = 1$ if we have

- (1) bb' is a block of Y ,
 (2) A is exactly the set of all symbols a in $\pi_b^{-1}(b)$ such that aa' is a block of X for some a' in A' .

Otherwise $M'_{AA'} = 0$. Consider the following subsets of the vertex set \mathcal{V} ,

$S = \{A \in \mathcal{V} : \text{there is a finite path in } G \text{ from } B \text{ to } A \text{ for some } B \in \mathcal{U}\},$

and

$S' = \{A \in \mathcal{V} : \text{there is a finite path in } G' \text{ from } A \text{ to } B \text{ for some } B \in \mathcal{U}\}.$

Theorem 3.1. *Using above notations, we have*

$$d_\pi = \min_{\substack{A \in S \\ A' \in S'}} \{|A \cap A'| : A \cap A' \neq \emptyset\}.$$

Proof. Note that G and G' are finite directed graphs. Let X_G and $X_{G'}$ be the shift spaces represented by G and G' accordingly. Let $\bar{\pi}_b$ be the map from \mathcal{V} to Y taking $A \in \mathcal{V}$ to $\pi(a)$ for some $a \in A$ (note that $\bar{\pi}_b(A)$ is independent of $a \in A$). Then $\bar{\pi}_b$ induces a one-block code $\bar{\pi}_G : X_G \rightarrow Y$ and a one-block code $\bar{\pi}_{G'} : X_{G'} \rightarrow Y$.

The key feature of these two graphs is the following. For any block

$$W = W_0 \dots W_k$$

of Y there is a unique walk

$$U = U_0 \dots U_k$$

in G with the following properties.

- (1) $U_0 = \pi_b^{-1}(W_0) = \bar{\pi}_b^{-1}(W_0).$
 (2) $\bar{\pi}_G(U) = W.$
 (3) $U_k = \{a \in \mathcal{A}(X) : \exists B \in \pi^{-1}(W) \text{ and } B_k = a\}.$

Similarly there is a unique walk

$$V = V_0 \dots V_k$$

in G' with the following properties.

- (1) $V_k = \pi_b^{-1}(W_k) = \bar{\pi}_b^{-1}(W_k).$
 (2) $\bar{\pi}_{G'}(V) = W.$
 (3) $V_0 = \{a \in \mathcal{A}(X) : \exists B \in \pi^{-1}(W) \text{ and } B_0 = a\}.$

Let $A \in S$, $A' \in S'$, and $A \cap A' \neq \emptyset$. Let $U = U_0 \dots U_c$ be a walk in G where $U_0 \in \mathcal{U}$ and $U_c = A$. Let $V = V_0 \dots V_k$ be a walk in G' where $V_0 = A'$ and $V_k \in \mathcal{U}$. Since $A \cap A' \neq \emptyset$ we have $\bar{\pi}_G(A) = \bar{\pi}_{G'}(A')$, and consequently $\bar{\pi}_G(U)\bar{\pi}_{G'}(V) = W_0 \dots W_{c+k} = W$ is a block of Y . By the key feature discussed in the above paragraph, we have

$$A \cap A' = \{a \in \mathcal{A}(X) : W' \in \pi_b^{-1}(W), \text{ and } W'_c = a\}.$$

Since by definition

$$d_\pi = \min_{\substack{W \text{ block of } Y \\ 0 \leq i \leq |W|}} |\{a \in \mathcal{A}(X) : \exists W' \in \pi^{-1}(W), W'_i = a\}|,$$

we have

$$d_\pi \leq \min_{\substack{A \in S \\ A' \in S'}} \{|A \cap A'| : A \cap A' \neq \emptyset\}.$$

Now we prove the inequality in the other direction; that is we seek to show that

$$d_\pi \geq \min_{\substack{A \in S \\ A' \in S'}} \{|A \cap A'| : A \cap A' \neq \emptyset\}.$$

Let $W = W_0 \dots W_k$ be a magic block of π with a magic coordinate $0 \leq c \leq k$. Let

$$D = \{a \in \mathcal{A}(X) : W' \in \pi_b^{-1}(W), W'_c = a\}.$$

Consider $U = W_0 \dots W_c$ and $V = W_c \dots W_k$. Let $\bar{U} = \bar{U}_0 \dots \bar{U}_c$ be the unique path in G which maps to U and $\bar{U}_0 = \pi_b^{-1}(W_0)$. Then

$$\bar{U}_C = \{a \in \mathcal{A}(X) : F \in \pi_b^{-1}(U), F_c = a\}.$$

Let $\bar{V} = \bar{V}_c \dots \bar{V}_k$ be a unique path in G' which maps to V and $\bar{V}_k = \pi_b^{-1}(W_k)$. Then

$$\bar{V}_c = \{a \in \mathcal{A}(X) : G \in \pi_b^{-1}(V), G_0 = a\}.$$

It follows that

$$\bar{U}_c \cap \bar{V}_c = \{a \in \mathcal{A}(X) : W' \in \pi_b^{-1}(W), W'_c = a\} = D.$$

Therefore

$$d \geq \min_{\substack{A \in S \\ A' \in S'}} \{|A \cap A'| : A \cap A' \neq \emptyset\},$$

which completes the proof. \square

4. CLASS DEGREE IS COMPUTABLE

In this section we find an upper bound on the length of a minimal transition block of a factor triple. Then by Theorem 2.10 it follows that the class degree of a factor code is computable. Without loss of generality, by Theorem 2.2 and Theorem 2.7 we may assume that the SFT is one-step and the factor code is one-block.

Theorem 4.1. *Let $\pi : X \rightarrow Y$ be a one-block factor code from a one-step SFT X to a sofic shift Y . Let $f = \max\{|\pi_b^{-1}(w)| : w \in \mathcal{A}(Y)\}$. There is a minimal transition block (W, n, M) of π with*

$$|W| \leq |\mathcal{A}(Y)| \times 2^{f^2+f+1}.$$

We need the following definitions and lemmas to prove Theorem 4.1.

Definition 4.2. *Let $\pi : X \rightarrow Y$ be a one-block factor code from a one-step SFT X to a sofic shift Y . Given $A, B \subseteq \mathcal{A}(X)$ and $\gamma \in \mathcal{P}(A \times B)$ say A **pairs with** B **in form** γ when there is a block $W = W_0 \dots W_n$ of Y with $\pi_b^{-1}(W)_0 = A$, $\pi_b^{-1}(W)_n = B$ such that $(a^*, b^*) \in \gamma$ if and only if there is $I \in \pi_b^{-1}(W)$ which begins at a^* and ends at b^* ; that is, $I_0 = a^*$ and $I_n = b^*$.*

Note that A can pair with B in at most $2^{|A \times B|}$ distinct forms.

Definition 4.3. *Let $\pi : X \rightarrow Y$ be a one-block factor code from a one-step SFT X to a sofic shift Y . Given $A, B \subseteq \mathcal{A}(X)$ and a block $W = W_0 \dots W_n$ of Y with $\pi_b^{-1}(W)_0 = A$, $\pi_b^{-1}(W)_n = B$, a **W -paring of** A, B , denoted by $P_W(AB)$, is the set $\gamma \in \mathcal{P}(A \times B)$ which contains all pairs (a^*, b^*) such that there is a block in $I \in \pi_b^{-1}(W)$ with $I_0 = a^*$ and $I_n = b^*$.*

Lemma 4.4. *Let $\pi : X \rightarrow Y$ be a one-block factor code from a one-step SFT X to a sofic shift Y . Let $A, B, C \subseteq \mathcal{A}(X)$. Let $D = D_0 \dots D_l$ and $D' = D'_0 \dots D'_{l'}$ be two blocks of Y with $\pi_b^{-1}(D)_0 = \pi_b^{-1}(D')_0 = A$, $\pi_b^{-1}(D)_l = \pi_b^{-1}(D')_{l'} = B$, and $P_D(AB) = P_{D'}(AB)$. Let $E = E_0 \dots E_s$ and $E' = E'_0 \dots E'_{s'}$ be two blocks of Y with $\pi_b^{-1}(E)_0 = \pi_b^{-1}(E')_0 = B$, $\pi_b^{-1}(E)_s = \pi_b^{-1}(E')_{s'} = C$, and $P_E(BC) = P_{E'}(BC)$. Then we have $P_W(AC) = P_{W'}(AC)$ where W and W' are blocks of Y formed by joining E to D and E' to D' as follows: $W = D_0 \dots D_l E_1 \dots E_s$ and $W' = D'_0 \dots D'_{l'} E'_1 \dots E'_{s'}$.*

Proof. First note that $D_l = E_0$ and $D'_{l'} = E'_0$ and therefore W and W' are legal blocks of Y with $|W| = l + s + 1$ and $|W'| = l' + s' + 1$. It is enough to show $P_W(AC) \subseteq P_{W'}(AC)$. Let (a^*, c^*) be in $P_W(AC)$. This means there is a block I in $\pi_b^{-1}(W)$ which starts at a^* and ends

at c^* . Since $B = \pi_b^{-1}(W)_l$ we have $I_l \in B$. Denote I_l by b^* . It follows that $(a^*, b^*) \in P_D(AB)$ and $(b^*, c^*) \in P_E(BC)$. Hence by assumption, $(a^*, b^*) \in P_{D'}(AB)$ and $(b^*, c^*) \in P_{E'}(BC)$. This means there is a block $U \in \pi_b^{-1}(D')$ with $U_0 = a^*$ and $U_{l'} = b^*$, and a block $V \in \pi_b^{-1}(E')$ with $V_0 = b^*$ and $V_{s'} = c^*$. Form the block J by joining the blocks V to U as below; $J = U_0 \dots U_{l'} V_1 \dots V_{s'}$. Clearly $J \in \pi_b^{-1}(W')$. Since $J_0 = a^*$ and $J_{l'+s'+1} = c^*$ it follows that $(a^*, c^*) \in P_{W'}(AC)$. \square

Lemma 4.5. *Let (W, n, M) be a transition block with $|W| = t + 1$. Let $\pi_b^{-1}(W)_0 = A$, $\pi_b^{-1}(W)_n = B$, and $\pi_b^{-1}(W)_t = C$ for some $A, B, C \subseteq \mathcal{A}(X)$. Let W' with $|W'| = t' + 1$ be another block of Y with $\pi_b^{-1}(W)_0 = A$, $\pi_b^{-1}(W)_{n'} = B$ for some $0 \leq n < t'$, and $\pi_b^{-1}(W)_{t'} = C$ such that $P_{W_0 \dots W_n}(AB) = P_{W'_0 \dots W'_{n'}}(AB)$ and $P_{W_n \dots W_t}(BC) = P_{W'_{n'} \dots W'_{t'}}(BC)$. Then (W', n', M) is a transition block of π .*

Proof. Note that by Lemma 4.4, since $P_{W_0 \dots W_n}(AB) = P_{W'_0 \dots W'_{n'}}(AB)$ and $P_{W_n \dots W_t}(BC) = P_{W'_{n'} \dots W'_{t'}}(BC)$, we have $P_W(AC) = P_{W'}(AC)$. Suppose $M = b_1, \dots, b_i$. Let $U \in \pi_b^{-1}(W')$, we need to show that U is routable through a member of M . Let $U_0 = a^*$ and $U_{t'} = c^*$. Since $(a^*, c^*) \in P_{W'}(AC)$ we have $(a^*, c^*) \in P_W(AC)$. It follows that there is a block $V \in \pi_b^{-1}(W)$ with $V_0 = a^*$ and $V_t = c^*$. Since (W, n, M) is a transition block, then block V must be routable through some symbol $b^* \in M$ which implies that $(a^*, b^*) \in P_{W_0 \dots W_n}(AB) = P_{W'_0 \dots W'_{n'}}(AB)$, and $(b^*, c^*) \in P_{W_n \dots W_t}(BC) = P_{W'_{n'} \dots W'_{t'}}(BC)$. It follows that there is a block $U' \in \pi_b^{-1}(W')$ with $U'_0 = a^*$, $U'_{n'} = b^*$, and $U'_{t'} = c^*$, meaning that U is routable through $b^* \in M$. \square

Proof of Theorem 4.1. Let (W, n, M) be a minimal transition block of π with $|W| = t + 1$. Assume $t + 1 > |\mathcal{A}(Y)| \times 2^{f^2+f+1}$. We construct a transition block (W', n', M') with $|W'| \leq |\mathcal{A}(Y)| \times 2^{f^2+f+1}$, but yet $M' = M$. Such transition block has depth $|M|$ and therefore is a minimal transition block.

Let $\pi_b^{-1}(W)_0 = A$, $\pi_b^{-1}(W)_n = B$. Denote $P_{W_0 \dots W_n}(AB)$ by γ . We show that A can pair with B in form γ in less than or equal $|\mathcal{A}(Y)| \times 2^{f^2+f}$ numbers of steps.

Recall that given any $d \in \mathcal{A}(Y)$ there are at most 2^f number of distinct subsets of $\pi_b^{-1}(d)$. Moreover, given $D \subseteq \pi_b^{-1}(d)$, A can pair with D in at most 2^{f^2} distinct forms. Therefore, since $n > |\mathcal{A}(Y)| \times 2^{f^2+f}$, there is at least one symbol $d \in \mathcal{A}(Y)$ and a subset $D \subseteq \pi_b^{-1}(d)$ such that d occurs in W at two distinct positions W_k and W_r , $k < r \leq n$, with $\pi_b^{-1}(W)_k = \pi_b^{-1}(W)_r = D$, and $P_{W_0 \dots W_k}(AD) = P_{W_0 \dots W_r}(AD)$. Let $Z = Z_0 \dots Z_s$ be the block $W_0 \dots W_k W_{r+1} \dots W_n$. Note that by

assumption, $W_k = W_r$ which implies that Z is a legal block of Y . If $s \leq |\mathcal{A}(Y)| \times 2^{f^2+f}$ then we are done, if not, we repeat the process until A pairs with D in form γ in less than $|\mathcal{A}(Y)| \times 2^{f^2+f}$ number of steps. The result is a block $G = G_0 \dots G_{n'}$ of Y where $n' \leq |\mathcal{A}(Y)| \times 2^{f^2+f}$, $\pi_b^{-1}(G)_0 = A$, $\pi_b^{-1}(G)_{n'} = B$, and $P_{G_0 \dots G_{n'}}(AB) = \gamma$.

Now let $\pi_b^{-1}(W)_t = C$, and denote $P_{W_n \dots W_t}(BC)$ by δ . By similar argument B can pair with C in less than or equal $|\mathcal{A}(Y)| \times 2^{f^2+f}$ number of steps; that is, there is a block $H = H_n \dots H_m$ of Y where $m - n + 1 \leq |\mathcal{A}(Y)| \times 2^{f^2+f}$, $\pi_b^{-1}(H)_n = B$, $\pi_b^{-1}(H)_m = C$, and $P_{H_n \dots H_m}(BC) = \delta$.

Make the new block $W' = W'_0 \dots W'_{t'}$ by joining the two blocks G and H as follows: $W' = G_0 \dots G_{n'} H_{n+1} \dots H_m$. Note that since $G_{n'} = H_n$ the block W' is a legal block of Y . Moreover, we have $|W'| \leq |\mathcal{A}(Y)| \times 2^{f^2+f+1}$. Now it is easy to see that W' is a minimal transition block. Note that $\pi_b^{-1}(W')_0 = A$, $\pi_b^{-1}(W')_{n'} = B$, $\pi_b^{-1}(W')_{t'} = C$, $P_{W'_0 \dots W'_{n'}}(AB) = \gamma$, and $P_{W'_{n'} \dots W'_{t'}}(BC) = \delta$. Lemma 4.5 implies that (W', n', M) is a transition block of π , and since its depth is $|M|$ we conclude that it is a minimal transition block of π . \square

5. OPEN QUESTION

Given a factor triple (X, Y, π) since $c_\pi \leq d_\pi$, if $d_\pi = 1$ then $c_\pi = 1$. Thus the algorithm given in Section 3 will compute the class degree as well. We can actually use the same two graphs G and G' and modify Theorem 3.1 by applying the upper bound on the length of a minimal transition block stated in Theorem 4.1 to write an algorithm to compute the class degree. However, since the bound on Theorem 4.1 is growing exponentially with respect to the number of preimages of a symbol in $\mathcal{A}(Y)$, the algorithm could be hopelessly complicated. It would be ideal to find an efficient algorithm for computing the class degree.

REFERENCES

- [1] M. Allahbakhshi and A. Quas. Class degree and relative maximal entropy. *Trans. Amer. Math. Soc.*, 365(3):1347–1368, 2013.
- [2] D. Blackwell. The entropy of functions of finite-state Markov chains. In *Transactions of the first Prague conference on information theory, Statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956*, pages 13–20. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957.
- [3] M. Boyle and S. Tuncel. Infinite-to-one codes and Markov measures. *Trans. Amer. Math. Soc.*, 285(2):657–684, 1984.

- [4] C. J. Burke and M. Rosenblatt. A Markovian function of a Markov chain. *Ann. Math. Statist.*, 29:1112–1122, 1958.
- [5] E. M. Coven and M. E. Paul. Finite procedures for sofic systems. *Monatsh. Math.*, 83(4):265–278, 1977.
- [6] D. Gatzouras and Y. Peres. Invariant measures of full dimension for some expanding maps. *Ergodic Theory Dynam. Systems*, 17(1):147–167, 1997.
- [7] K. H. Kim and F. W. Roush. An algorithm for sofic shift equivalence. *Ergodic Theory Dynam. Systems*, 10(2):381–393, 1990.
- [8] D. Lind and B. Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995.
- [9] K. Petersen. Information compression and retention in dynamical processes. In *Dynamics and randomness (Santiago, 2000)*, volume 7 of *Nonlinear Phenom. Complex Systems*, pages 147–217. Kluwer Acad. Publ., Dordrecht, 2002.

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